# A MORE EXPLICIT SOLUTION OF THE LONG-ONLY MINIMUM VARIANCE OPTIMIZATION PROBLEM

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# Min Variance Long-Only Optimization Problem

- Markowitz [1952] pioneered Portfolio Optimization and now we want to solve the long-only problem

$$\begin{aligned} \min_{\mathbf{w}\in\mathbf{R}^{\mathbf{p}}} \quad & \mathbf{w}^\top \Sigma \mathbf{w} \text{ s.t} \\ & \mathbf{w}^\top \mathbf{1} = 1 \\ & w_i \geq 0 \end{aligned} \tag{1}$$

# Covariance Matrix from the One-factor Model

- Assumption of a single-factor returns model
- Spiked covariance matrix $\Sigma = [\sigma_{ij}; i, j \in P] = \sigma^2 \beta \beta^\top + diag(\delta_e^2)$
- where $\sigma^2$ is the market risk in terms of variance
- $\beta$ is the vector of the market factor weightings
- $\delta_e^2 = [\delta_1^2, \delta_2^2, ..., \delta_p^2]$ is the vector $\in \mathbf{R}^p$ of idiosyncratic risk of each asset.

# Motivation

- In reality, $\Sigma$ is unknown.
- We want to use the James-Stein for Eigenvectors estimation method to estimate $\beta$.
- The JSE estimation method has been shown to better the PCA method in the long-short portfolio.
- The JSE method requires the use of an explicit solution of the problem.

# The Implicit Solution

- The current widely accepted solution for the long-only problem is found in Clarke et al [2011]

$$w_i = \frac{\sigma^2_{LMV}}{\delta_i^2}\left(1 - \frac{\beta_i}{\beta^L}\right) \text{ for } \beta_i < \beta^L \text{ else } w_i = 0 \tag{2}$$

where the threshold beta is defined by

$$\beta^L = \frac{\frac{1}{\sigma^2} + \sum_{\beta_i < \beta^L} \frac{\beta_i^2}{\delta_i^2}}{\sum_{\beta_i < \beta^L} \frac{\beta_i}{\delta_i^2}} \tag{3}$$

where

$$\sigma^2_{LMV} = \frac{1}{\sum_{\beta_i < \beta^L} \frac{1}{\delta_l^2}\left(1 - \frac{\beta_l}{\beta^L}\right)}$$

is described in Clarke et al[2011] as the ex-ante return variance of the long-only minimum-variance portfolio.

# Assumptions

**THE KEY QUESTION**

Assuming $\beta, \sigma^2, \delta_e^2$ known, can we find an explicit solution to the optimization problem (1) for the one-factor model such that we can use the solution in our JSE estimation?

1. The betas are ordered increasingly, i.e $i < j \implies \beta_i \leq \beta_j$
2. The betas are positive. Generally, assets move in the direction of the market.

# Observation from the Implicit Solution

**Lemma**

If all the $p$ assets are ordered according to assumption 1, then there exists a $k \leq p$ such that

1. $w_i > 0 \ \forall \ i \leq k$.
2. $w_i = 0 \ \forall \ i > k$.

i.e $\mathbf{w} = [w_i > 0, w_2 > 0, ..., w_k > 0, w_{k+1} = 0, w_{k+2} = 0, ..., w_p = 0]$

# The Explicit Solution of the long-only constrained problem

## Proposition

Given the assumptions, and let $k$ be as defined, then the non-zero components of the solution $\mathbf{w} = [w_1, w_2, ..., w_p]$ of the long-only problem (1) over $\mathbf{R}^p$ is equivalent to the solution of the global problem over the assets $i \leq k$

$$\min_{w^* \in \mathbf{R}^k} {w^*}^\top \Sigma^* w^* \quad {w^*}^\top \mathbf{1} = 1 \tag{4}$$

where $\mathbf{w}^* = [w_1, w_2, ..., w_k]$, and $\Sigma^*$ is the $k \times k$ sub-matrix of $\Sigma$ consisting of its first $k$ rows and columns.

# The Solution of the Long-Only Constrained Problem

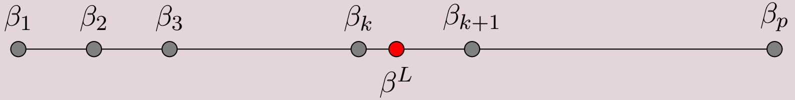
## Takeaway from Proposition 1

- if $k$ is known, then active assets in the long only portfolio is known.
- Explicit formula for the weight is:

$$\mathbf{w}^* = \frac{(\Sigma^*)^{-1}\mathbf{1}^*}{\mathbf{1}^{*T}(\Sigma^*)^{-1}\mathbf{1}^*} \tag{5}$$

- All other assets are assigned a weight of 0.
- Can we get an explicit definition of $k$?

## The Ordered Betas



# The Necessary Threshold: Summary

$$\Sigma = \begin{bmatrix}
\sigma^2\beta_1^2+\delta_1^2 & \sigma^2\beta_1\beta_2 & \dots & \sigma^2\beta_1\beta_k & \sigma^2\beta_1\beta_{k+1} & \dots & \sigma^2\beta_1\beta_p \\
\sigma^2\beta_1\beta_2 & \sigma^2\beta_2^2+\delta_2^2 & \dots & \sigma^2\beta_2\beta_k & \sigma^2\beta_2\beta_{k+1} & \dots & \sigma^2\beta_2\beta_p \\
. & . & \dots & . & \dots & . & \\
. & . & \dots & . & \dots & . & \\
. & . & \dots & . & \dots & . & \\
\sigma^2\beta_1\beta_k & \sigma^2\beta_2\beta_k & \dots & \sigma^2\beta_k^2+\delta_k^2 & \sigma^2\beta_k\beta_{k+1} & \dots & \sigma^2\beta_k\beta_p \\
. & . & \dots & . & \dots & . & \\
. & . & \dots & . & \dots & . & \\
. & . & \dots & . & \dots & . & \\
\sigma^2\beta_1\beta_p & \sigma^2\beta_2\beta_p & \dots & \sigma^2\beta_p\beta_k & \sigma^2\beta_p\beta_{k+1} & \dots & \sigma^2\beta_p^2+\delta_p^2
\end{bmatrix}$$

# The Necessary Threshold: Summary

$$\sigma^2 \beta_i \beta_j < \sigma^2 \beta_j \beta_k \ \forall i, j < k$$

$$\sigma^2 \beta_j \beta_k < \sigma^2 \beta_j^2 + \delta_j^2 \ \forall j < k$$

The equations above imply that

$$\sigma^2 \beta_i \beta_j < \sigma^2 \beta_j^2 + \delta_j^2 \ \forall i, j \leq k$$

Therefore

$$i \leq k \implies \beta_i < \min_{j \in P} \{\beta_j + \frac{\delta_j^2}{\sigma^2 \beta_j}\}$$

# The Necessary Threshold: Derivation

- $i \leq k \implies w_i > 0.$
- $w_i > 0 \implies \frac{1}{\sigma^2} + (\sum_{j=1}^{k} \frac{\beta_j^2}{\delta_j^2}) - \sum_{j=1}^{k} \frac{\beta_i \beta_j}{\delta_j^2} > 0$
- We switch the focus to $i = k$ and fix an arbitrary $j' \leq k$
-
$$\frac{1}{\sigma^2} + \frac{\beta_{j'}^2}{\delta_{j'}^2} - \frac{\beta_k \beta_{j'}}{\delta_{j'}^2} > \sum_{j=1, j \neq j'}^{k-1} \left( \frac{\beta_j}{\delta_j^2} (\beta_k - \beta_j) \right) > 0 \tag{6}$$
-
$$\frac{1}{\sigma^2} + \frac{\beta_j^2}{\delta_j^2} - \frac{\beta_k \beta_j}{\delta_j^2} > 0 \; \forall \; j < k \tag{7}$$
-
$$\beta_k < \beta_j + \frac{\delta_j^2}{\sigma^2 \beta_j} \; \forall \; j < k \tag{8}$$

# The Necessary Threshold: Derivation cont.

- $$i \leq k \implies \beta_i < \beta_j + \frac{\delta_j^2}{\sigma^2 \beta_j} \; \forall \; j \leq k \tag{9}$$

- $$\beta_i < \beta_j < \beta_j + \frac{\delta_j^2}{\sigma^2 \beta_j} \; \forall \; j > k \tag{10}$$

- $$i \leq k \implies \beta_i < \beta_j + \frac{\delta_j^2}{\sigma^2 \beta_j} \; \forall \; j \leq p \tag{11}$$

  Therefore we can conclude that

- $$i \leq k \implies \beta_i < \min_{j \leq p}\{\beta_j + \frac{\delta_j^2}{\sigma^2 \beta_j}\} \tag{12}$$

# The Necessary Threshold

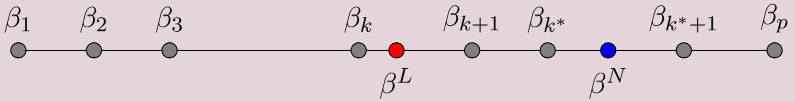
**Proposition**

Under Assumptions 1 and 2, define $\beta^N = \min_{j \leq p}\{\beta_j + \frac{\delta_j^2}{\sigma^2 \beta_j}\}$. If $i \leq k$, then $\beta_i < \beta^N$.

- Caveat: $\beta_i < N_t$ is a necessary but not sufficient condition for assets to have positive weights ($w_i > 0$). We showed that in some situations it is possible to find an asset $i$ with $\beta_i < \beta^N$ but $i > k$.
- Define $k^* = \max\{i \in P; \beta_i < \beta^N\}$.

# The Necessary Threshold $\beta^N$

## The Ordered Betas



# Experiment: The Usefulness of $\beta^N$

- $p = 1000$
- Simulation: Positive betas with mean 1.1 and variance 0.6
- Assets with $\beta_i < \beta^N$: 248
- Assets immediately discarded from consideration: 752

# Sufficient Threshold: Focus on assets $k$ and $k+1$

$w_k > 0$ and $w_{k+1} = 0$

$$w_k > 0 \implies \beta_k < \frac{\frac{1}{\sigma^2} + (\sum_{j=1}^{k-1} \frac{\beta_j^2}{\delta_j^2})}{\sum_{j=1}^{k-1} \frac{\beta_j}{\delta_j^2}} \tag{13}$$

Recalling the KKT conditions, we see that

$$w_{k+1} = 0 \implies \beta_{k+1} \geq \frac{\frac{1}{\sigma^2} + (\sum_{j=1}^{k} \frac{\beta_j^2}{\delta_j^2})}{\sum_{j=1}^{k} \frac{\beta_j}{\delta_j^2}} \tag{14}$$

we note that for every other asset $i \geq k+1$, (14) holds.

# Sufficient Threshold: Focus on assets $k$ and $k+1$

Comparing Equations (13) and (14), $k$ is the maximum $i \leq k^*$ such that the difference

$$\frac{\frac{1}{\sigma^2} + \left(\sum_{j=1}^{i-1} \frac{\beta_j^2}{\delta_j^2}\right)}{\sum_{j=1}^{i-1} \frac{\beta_j}{\delta_j^2}} - \beta_i \tag{15}$$

is positive

# The Sufficient Threshold

## Proposition

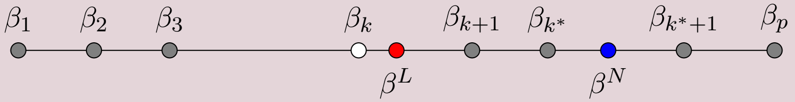
(The Sufficient Threshold) Under assumptions 1 and 2, the sufficient condition for an asset $i$ to have positive weight in the long-only portfolio is $\beta_i \leq \beta_k$, where

$$k = \max\left\{ i \leq k^* : \frac{\frac{1}{\sigma^2} + \sum_{j=1}^{i-1} \frac{\beta_j^2}{\delta_j^2}}{\sum_{j=1}^{i-1} \frac{\beta_j}{\delta_j^2}} - \beta_i > 0 \right\} \tag{16}$$

and $k^* = \max\{i \in P; \beta_i < \beta^N\}$.

# The Sufficient Threshold $\beta_k$

## The Ordered Betas



# Theorem 4: (The Explicit Solution to the long-only constrained problem under the 1-factor model)

Assuming that the betas are positive and ordered increasingly, the solution to the long only constrained optimization problem (1) under the 1-factor model assumptions is given as

$$w^* = \frac{(\Sigma^*)^{-1}\mathbf{1}^*}{\mathbf{1}^{*T}(\Sigma^*)^{-1}\mathbf{1}^*} \text{ which gives } w_i \text{ for } i\text{: } i = 1, ..., k \qquad (17)$$
$$w_i = 0 \text{ for } i > k$$

$\mathbf{w}^* = [w_1, w_2, ..., w_k]$, $\Sigma^*$ is the $k \times k$ sub-matrix of $\Sigma$ and $k$ is defined as

$$k = \max\left\{ i \leq k^* : \frac{\frac{1}{\sigma^2} + \sum_{j=1}^{i-1} \frac{\beta_j^2}{\delta_j^2}}{\sum_{j=1}^{i-1} \frac{\beta_j}{\delta_j^2}} - \beta_i > 0 \right\} \qquad (18)$$

with $k^* = \max\{i \in P; \beta_i < \beta^N\}$ and $\beta^N = \min_{j \in P}\{\beta_j + \frac{\delta_j^2}{\sigma^2 \beta_j}\}$.

# Thank you

Thank you for your time !!! Questions?

# REFERENCES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